Sponsored by Sandia National Laboratories **November 15, 2003**

Go Figure 2003

For Students in grades 7, 8, 9, 10, 11, and 12

Show your work. You can receive partial credit for partial solutions. Please write all solutions clearly, concisely, and legibly.

The positive integers are the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 \ldots$

1. (a) Jane invested \$1000. Her investment decreased in value by 10% over two years. How much was the investment worth after two years?

   (b) Jack invested \$1000. His investment decreased by 5% in the first year and it decreased by 5% in the second year. How much was the investment worth after two years?

2. In this problem, each of A, B and C represents a different digit. For example, if F represents 6 then $3F$ represents 36 and $F4F$ represents 646. In each of these problems, the hidden digit is different from all other digits used in the problem. For example, in the first problem, none of A, B, or C is in the set $\{3, 6, 7, 9\}$. For each of the following problems, find digits A, B, and C that make the problem a correct statement.

   (a) $7A3 \times BA = C9AB6$.

   (b) $\frac{A39}{C9B} = \frac{A3}{CB} = \frac{A}{B}$.

3. (a) How many ways can you make \$510 from only 20 and 50 dollar bills?

   (b) A box contains some coins, not necessarily of all types. For example, there may be no pennies. You cannot make change for a dollar from the contents (no subset of the coins has value exactly \$1). What is the maximum total value of the set of coins in the box?

4. A book is lying open on a table with two pages visible. When the page numbers on these two pages are multiplied, the resulting product is 7656. What are the numbers on these two pages?

5. In an arithmetic progression, the difference $s - t$ of adjacent terms t, s is fixed. For example, the arithmetic progression $4, 8, 12, 16 \ldots$ has $(8 - 4) = (12 - 8) = (16 - 12) = 4$ as the fixed difference.

   (a) Find the 59th term of the arithmetic progression $5, 10, 15, \ldots$ [The three dots after 15 are read as "and so on" to indicate that this progression continues with the pattern required of an arithmetic progression without end.].

   (b) How many terms are in the arithmetic progression $12, 17, 22, \ldots, 322$?

   (c) What is the sum of the first and last terms of the arithmetic progression $8, 13, 18, \ldots, 423$?

   (d) How many (unordered) pairs of terms in the arithmetic progression of part (c) sum to exactly 431?

   (e) What is the sum of all the terms of the arithmetic progression of part (c)?

   (f) What is the maximium number of terms can we select from the arithmetic progression of part (c) such that no pair of selected numbers sums to exactly 446?

6. In a geometric progression, the ratio s/t of adjacent terms t, s is fixed. For example, the geometric progression $2, 4, 8, 16 \ldots$ has $(16/8) = (8/4) = (4/2) = 2$ as the fixed ratio. Find the missing terms (represented by letters) in the following geometric sequences.

   (a) $1, a, 9, 27, b$

   (b) $4, 6, 9, c, d$

(c) $8, 4, 2, e, f$

(d) $64, 48, 36, g, h$

7. Define $S(n) = 1+2+\ldots+2^n$, where 2^n means 2 multiplied by itself n times. For example $2^5 = 2\times2\times2\times2\times2 = 32$. Therefore $S(1) = 1 + 2 = 3$ and $S(4) = 1 + 2 + 4 + 8 + 16$.

   (a) What is $S(1) + 1$? $S(2) + 1$? $S(3) + 1$? What is $S(4) + 1$?

   (b) What is $2^5 + 2^6 + 2^7 + 2^8 + 2^9$? (Hint recall $S(4) = 1 + 2 + 2^2 + 2^3 + 2^4$.)

   (c) What is $S(10) + 1$?

8. The greatest common divisor of two numbers n and m, denoted $\mathrm{GCD}(n, m)$ is the largest number that divides both n and m evenly. The least common multiple of two numbers n and m, denoted $\mathrm{LCM}(n, m)$, is the smallest number that is a multiple of both numbers. For example, $\mathrm{GCD}(2, 3) = 1$, $\mathrm{LCM}(2, 3) = 6$, $\mathrm{GCD}(5, 25) = 5$, $\mathrm{LCM}(5, 25) = 25$. The notation x^y means x multiplied by itself y times. For any x, we define $x^0 = 1$.

   (a) What is $\mathrm{GCD}(10, 15)$ and $\mathrm{LCM}(10, 15)$?

   (b) What is $\mathrm{GCD}(12, 18)$ and $\mathrm{LCM}(12, 18)$? Note that $12 = 2^2 \times 3$ and $18 = 2 \times 3^2$.

   (c) Let $n = 2^{400}3^{56}5^{102}7^{33}11^{25}$ and $m = 2^{81}3^{10}5^{270}7^{86}13^2$. Find values of the exponents (represented as letters) that make the following statements true: $\mathrm{GCD}(n, m) = 2^a3^b5^c7^d11^e13^f$, and $\mathrm{LCM}(n, m) = 2^g3^h5^i7^j11^k13^l$.

   (d) How many (unordered) pairs of numbers have 6 as their greatest common divisor and 2100 as their least common multiple?

9. A permutation on the numbers $1, 2, \ldots, n$ changes the order of these numbers. We represent a permutation by listing the reordering. For example $(4, 6, 3, 1, 5, 2)$ is a permutation of $1, \ldots, 6$. We can use permutations to show the change in order of a sequence of elements which may be numbers, letters, or characters as represented in a table. For example, suppose we are given the sequence of elements (s, h, w, i). The table

1	2	3	4
s	h	w	i

indicates that "s" is the first letter, "h" is the second, "w" is the third, and "i" is the fourth. Applying the permutation $(3, 4, 1, 2)$ to a sequence of four characters means to put the third letter first, the fourth letter second, then the first letter, then the second. Applying this permutation to the table gives:

3	4	1	2
w	i	s	h

. For this problem, we say a permutation is collapsed if all the commas separating the elements (and parentheses surrounding them) are removed, running all the elements together. For example, $(3, 6, 2, 12, 5, 10, 9, 1, 4, 7, 11, 8)$ is the representation of a permutation on $1, \ldots, 12$. The collapsed representation is 362125109147118.

We are given 27 elements as shown in the following table:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
V	A	E	U	D	E	E	Z	E	U	S	O	L	P	O	H	V	S	U	L	Z	T	M	R	O	L	Y

This collapsed permutation of the above table contains a message with multiple words (with no spaces between the words):

$$232224177261541127251018121313522166141921 8209$$

Determine where to place commas to recover the numbers $1, \ldots, 27$, apply the permutation to the table and unscramble the message.

10. If n is a positive integer then $\sigma(n)$ is the sum of all its divisors. For example, $\sigma(10) = 1 + 2 + 5 + 10$.

    (a) What is $\sigma(3), \sigma(5)$, and $\sigma(15)$?

(b) What is $\sigma\left(2^{10}\right)$?

(c) $2^{11}-1$ is prime. What is $\sigma(2^{11}-1)$?

(d) What is $\sigma\left(2^{10}\left(2^{11}-1\right)\right)$? You may leave this as a product of terms rather than calculating a single number.

11. Consider the set of positive numbers with no repeated digits in a single number:

$$\{1, 2, \ldots, 9, 10, 12, 13, \ldots, 20, 21, 23, \ldots, 9876543210\}$$

ordered from smallest to largest.

(a) How many of these numbers have six digits?

(b) How many of these numbers have six or fewer digits?

(c) What is the 288657-th number in the list? (Check: If the digits 1 through 9 are replaced with characters $A, B, \ldots, I$ respectively, then your answer should spell 2 English words).

12. As clearly as you can, justify your answer for Problem 11c.

PROBLEM 13 IS OPTIONAL FOR STUDENTS IN GRADES 7, 8, AND 9.

13. $ACEG$ is a square with side length 1. The square is folded in half along the line segment BF and then reopened. That is, B is the midpoint of segment AC and F is the midpoint of segment GE. The square is then folded upward along segment DH until the corner E meets point B. Therefore quadrilateral $HIBD$ is the reflection of quadrilateral $GHDE$ about the line DH.

(a) What is the length of segment CD?

(b) What is the length of segment GJ?